6.1 Reciprocal, Quotient and Pythagorean Identities

Goals:

- Verifying a trigonometric identity numerically and graphically using technology.
- Exploring reciprocal, quotient and Pythagorean identities.
- Determining non-permissible values of trigonometric identities.
- Explaining the difference between a trigonometric identity and a trigonometric equation.

(I) Remembering Reciprocal and Quotient Identities

* Have to remember

Primary Trig Ratios:

$$\csc\theta = \frac{1}{\sin\theta} \qquad \sec\theta = \frac{1}{\cos\theta} \qquad cot\theta = \frac{1}{tan\theta}$$

Quotient Trig Ratios:

$$tan\theta = \frac{sin\theta}{cos\theta} \qquad cot\theta = \frac{cos\theta}{sin\theta}$$

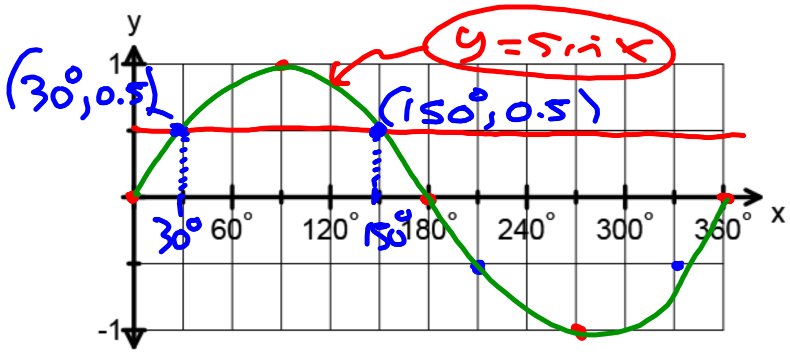
(II) Explaining the Difference between a Trig Identity and Trig Equation

(A) Solving a Trig Equation

Example: Let $y = \sin x$ Let $y = \frac{1}{2}$

Solve the equation $\sin x = \frac{1}{2}$ graphically over the domain $0° \le x \le 360°$.

Graph the equations $y = \sin x$ and $y = \frac{1}{2}$.



Solutions for x: 30°, 150°

When we solve a **trig equation**:

- visually it represents the intersection points.
- The solutions are a small set of domain (or x) values.

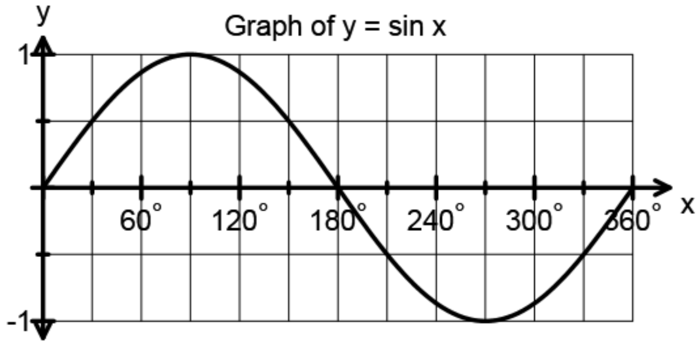
(B) Verifying a Trig Identity

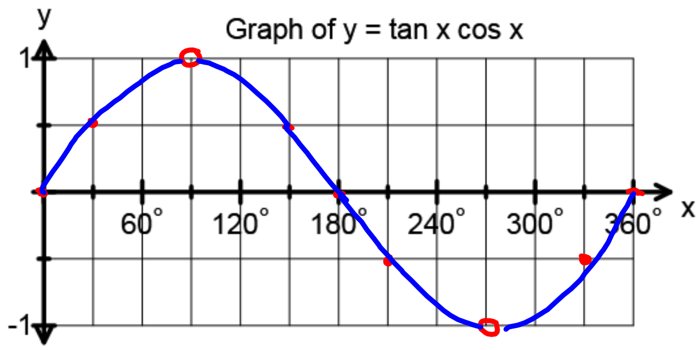
Example:
Determine graphically the validity of the trig identity:

Let $y = \sin x$ Let $y = \tan x \cos x$

$$\boxed{\sin x} = \boxed{\tan x \cos x}$$

Graph $y = \sin x$ and $y = \tan x \cos x$
over the domain $0° \leq x \leq 360°$.





What points represent the only notable differences between the two graphs? at 90° and 270°

We can conclude that $\sin x = \tan x \cos x$ is an identity since the expressions are equivalent for all permissible values.

where we have defined behaviour for each domain value

When we solve a **trig identity** such as **sin x = tan x cos x**

- visually it represents all the permissible values of the variable on both sides of the equation.

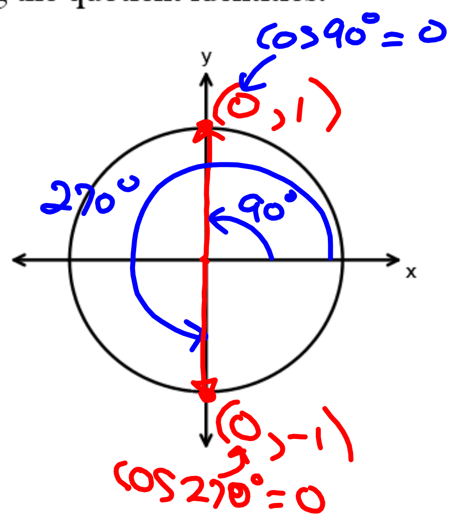
Why are there non – permissible values for y = tan x cos x ?

Rewrite the function y = tan x cos x using the quotient identities.

$y = \dfrac{\sin x}{\boxed{\cos x}} \cdot \cos x$

non permissible values are attained when $\cos x = 0$

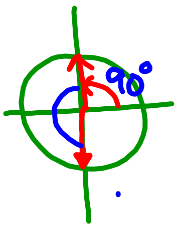
x-coordinate of pt on unit circle → x = 0? where?



Trig Identities:

- Are undefined when we have division by 0.

Write a general expression that represents all of the non – permissible values for: sin x = tan x cos x.



NPV's

$\dfrac{\sin x}{\cos x} \cos x$

$x = 90° + 180° n,\ n \in I$

or

$x = \frac{\pi}{2} + \pi n,\ n \in I$

Non – permissible values often occur when trig expressions contain:

- A rational expression where a denominator value is __0__.
- Tangent, cotangent, secant and cosecant since these expressions contain non – permissible values in their domain.

(III) Verifying a Trig Identity Graphically and Numerically

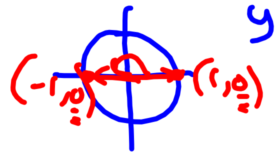
Example:

Given $\cot x = \dfrac{\cos x}{\sin x}$

(a) Determine the non – permissible values in degrees.

$\cot x = \dfrac{\cos x}{\sin x}$ — where is $\sin x = 0$? $y = 0$?

NPV's $\boxed{x \neq 180^\circ n, n \in I}$

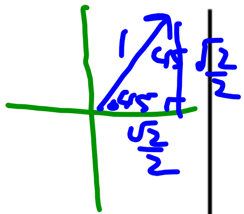


(b) Verify that x = 45° and $x = \frac{\pi}{6}$ are solutions to the equation.

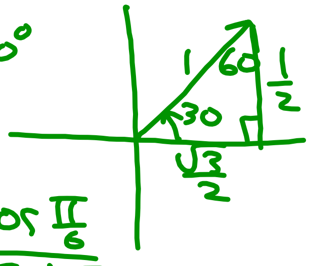
When $x = 45^\circ$

$\underbrace{\cot 45^\circ}$ $= 1$

$\dfrac{\cos 45^\circ}{\sin 45^\circ} = \dfrac{\frac{\sqrt{2}}{2}}{\frac{\sqrt{2}}{2}} = 1$



$x = \frac{\pi}{6} = 30^\circ$



$\underbrace{\cot \frac{\pi}{6}}$ $\frac{\text{adj}}{\text{opp}}$

$\dfrac{\sqrt{3}/2}{1/2}$

$\frac{\sqrt{3}}{\cancel{2}} \times \cancel{2}$

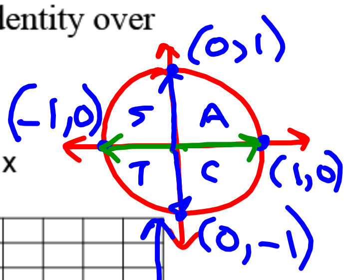
$\sqrt{3}$

$\dfrac{\cos \frac{\pi}{6}}{\sin \frac{\pi}{6}}$

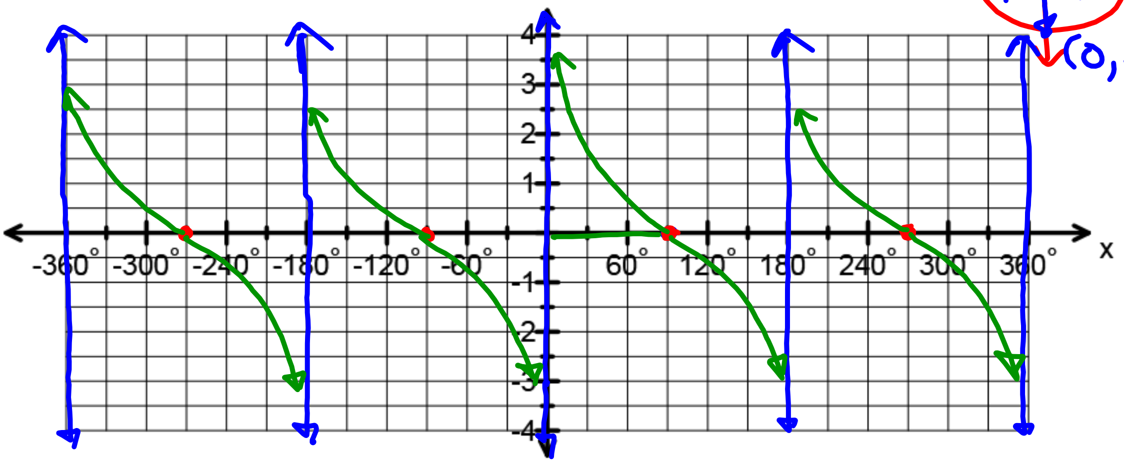
$\dfrac{\frac{\sqrt{3}}{2}}{1/2}$

$\therefore \sqrt{3}$

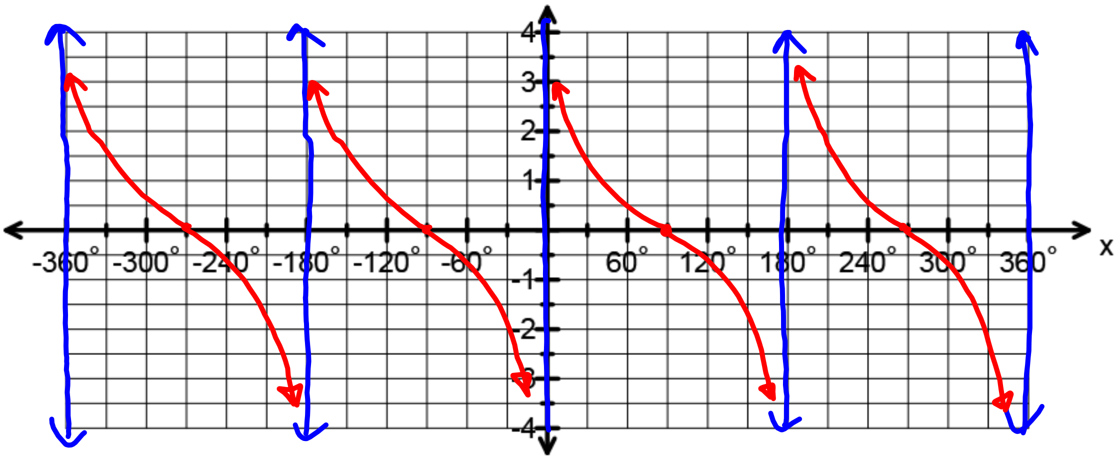
(c) Graphically decide whether the equation is an identity over the domain $-360° \le x \le 360°$.



Graph of y = cot x



Graph of $y = \frac{\cos x}{\sin x}$

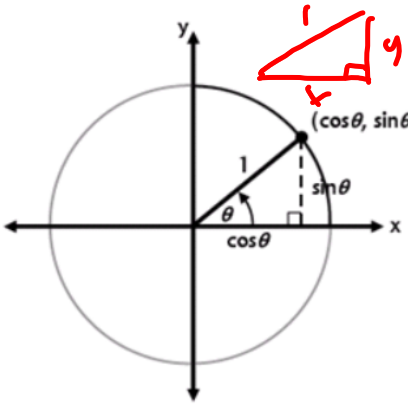


Conclusion: The graphs are the same since they both have the same NPU's

(IV) Pythagorean Identities

Remember points on the terminal arm of Θ

Since x = cos θ and y = sin θ



Pythagorean Theorem

$x^2 + y^2 = 1$

$[\cos\theta]^2 + [\sin\theta]^2 = 1$

$$\boxed{\cos^2\theta + \sin^2\theta = 1}$$

We can use the Pythagorean Identity:

to derive other Pythagorean Identities.

Remember

① $\tan^n x = \dfrac{\sin^n x}{\cos^n x}$

$\boxed{n \in N}$

② $\sec^n x = \dfrac{1}{\cos^n x}$

③ $\csc^n x = \dfrac{1}{\sin^n x}$

④ $\cot^n x = \dfrac{\cos^n x}{\sin^n x}$

⑤ $\cot^n x = \dfrac{1}{\tan^n x}$

(a) Multiply $\cos^2x + \sin^2x = 1$ by $\boxed{\dfrac{1}{\cos^2x}}$ NPV $x \neq 90° + 180°n, n \in I$

where is $\cos x = 0$

(0,1) (0,−1)

$\dfrac{1}{\cancel{\cos^2 x}} \cdot \cancel{\cos^2 x} + \dfrac{1}{\cos^2 x} \cdot \sin^2 x = 1 \cdot \dfrac{1}{\cos^2 x}$

$1 + \dfrac{\sin^2 x}{\cos^2 x} = \dfrac{1}{\cos^2 x}$

$$\boxed{1 + \tan^2 x = \sec^2 x}$$

(b) Multiply $\cos^2x + \sin^2x = 1$ by $\boxed{\dfrac{1}{\sin^2x}}$ NPV $x \neq 180°n, n \in I$

$\dfrac{1}{\sin^2 x} \cdot \cos^2 x + \dfrac{1}{\cancel{\sin^2 x}} \cdot \cancel{\sin^2 x} = \dfrac{1}{\sin^2 x} \cdot 1$

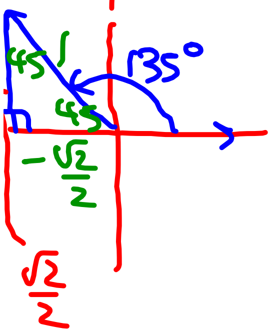
$\dfrac{\cos^2 x}{\sin^2 x} + 1 = \dfrac{1}{\sin^2 x} \Rightarrow \boxed{\cot^2 x + 1 = \csc^2 x}$

(V) Numerically Verifying Pythagorean Identities

(a) Verify that the equation: $\underbrace{1 + \tan^2 x}_{\text{LHS}} = \underbrace{\sec^2 x}_{\text{RHS}}$

is numerically true for $x = \frac{3\pi}{4}$.

$x = \frac{3\pi}{4} = 135^\circ$



Left Hand Side — Right Hand Side

LHS	RHS
$1 + \tan^2 x$	$\sec^2 x$
$1 + \left(\frac{\sqrt{2}/2}{-\sqrt{2}/2}\right)^2$	$\left(-\frac{2}{\sqrt{2}}\right)^2$
$1 + (-1)^2$	$\frac{4}{2}$
$1 + 1$	2

2

(b) Verify that the equation: $\cot^2 x + 1 = \csc^2 x$

is numerically true for $x = \frac{\pi}{6}$.

LHS	RHS

(VI) Using Reciprocal, Quotient and Pythagorean Identities to Simplify Trigonometric Identities

Strategy for Simplifying Trig Identities:

- Write the expression in terms of sin x or cos x using Reciprocal, Quotient or Pythagorean Identities

Primary Trig Ratios:

$\csc\theta = \frac{1}{\sin\theta}$ $\sec\theta = \frac{1}{\cos\theta}$ $\cot\theta = \frac{1}{\tan\theta}$

Quotient Trig Ratios:

$\tan\theta = \frac{\sin\theta}{\cos\theta}$ $\cot\theta = \frac{\cos\theta}{\sin\theta}$

- **Pythagorean Identities** can be manipulated and used to help simplify an expression

$\boxed{\cos^2 x + \sin^2 x = 1}$	$1 + \tan^2 x = \sec^2 x$	$\cot^2 x + 1 = \csc^2 x$
* $\cos^2 x = 1 - \sin^2 x$	$\boxed{\tan^2 x = \sec^2 x - 1}$	$\cot^2 x = \csc^2 x - 1$
* $\sin^2 x = 1 - \cos^2 x$	$1 = \sec^2 x - \tan^2 x$	$1 = \csc^2 x - \cot^2 x$

Solve for $\cos^2 x$

Solve for $\sin^2 x$

- Execute the algebraic operations to simplify the expression

Example: Determine non – permissible values and simplify the expression $\dfrac{\sec x}{\tan x}$

① Change everything into sin x and cos x using quotient identities

② Identify all NPV's

③ Simplify by executing the algebra

$$= \dfrac{\frac{1}{\cos x}}{\frac{\sin x}{\cos x}}$$

$$= \dfrac{1}{\cancel{\cos x}} \times \dfrac{\cancel{\cos x}}{\sin x}$$

$$= \dfrac{1}{\sin x}$$

$$= \csc x$$

$$\dfrac{\frac{a}{b}}{\frac{c}{d}} \div \quad = \dfrac{a}{b} \div \dfrac{c}{d}$$

$\cos x = 0$?

$x \neq 90° + 180°n, n \in I$

$\sin x = 0$?

$x \neq 180°n, n \in I$

p.296 #3, #4

*Example: Determine non – permissible values and simplify the expression $\dfrac{\cot x}{\csc x \cos x}$

I Express in terms of sin x and cos x

II NPV's same as above

$$= \dfrac{\frac{\cos x}{\sin x}}{\frac{1}{\sin x}\cos x}$$

$$= \dfrac{\cancel{\cos x}}{\cancel{\sin x}} \cdot \dfrac{\cancel{\sin x}}{\cancel{\cos x}}$$

$$= 1$$

Example: Determine non – permissible values and simplify the expression $\frac{\csc x - \sin x}{\cot^2 x}$

$= \dfrac{\dfrac{1}{\sin x} - \dfrac{\sin x}{1}\sin^2 x}{\dfrac{\cos^2 x}{\sin^2 x}\sin^2 x}$

(I) Identify an LCD for all of the fractions
LCD $= \sin^2 x$

GCF $= \sin x$

$= \dfrac{\sin x - \sin^3 x}{\cos^2 x}$

(II) Multiply the LCD to the numerator of every fraction to eliminate the denominators

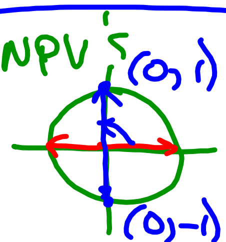
(III) consider any possible Factoring
GCF = ?

$= \dfrac{\sin x\,(1-\sin^2 x)}{\cos^2 x}$

$= \dfrac{\sin x\,\cancel{\cos^2 x}}{\cancel{\cos^2 x}}$

$= \underline{\sin x}$

NPV's



$x \neq 180^\circ n, n \in I$

$x \neq 90^\circ + 180^\circ n, n \in I$

(IV) Are there any identities we can use to help simplify

P.296 – P.298 #1 – #6, #8, #10, #11 – #14, #16, C1, C2

Worksheet – Simplifying Trig Identities

Determine the NPV's and simplify each identity:

NPV's
none

1. $(1 + \sin x)(1 - \sin x)$

2. $\tan\theta\cot\theta + \cos\theta\sec\theta$

3. $(\csc\theta - 1)(\csc\theta + 1)$

4. $\dfrac{\sin B}{\csc B} + \dfrac{\cos B}{\sec B}$

5. $\frac{1+\tan^2\theta}{1+\cot^2\theta}$

6. $\frac{tanx+cotx}{\sec^2 x}$

7. $(\csc A - \cot A)(1 + \cos A)$

NO NPV's 8. $\underbrace{\sin^3\theta + sin\theta\cos^2\theta}$ FACTOR → GCF = $\sin\theta$

$= \sin\theta\left[\underbrace{\sin^2\theta + \cos^2\theta}_{1}\right]$

$= \sin\theta$

ANSWERS:

1. No NPV's $\cos^2 x$ 2. NPV's $\theta \neq \frac{\pi}{2} + \pi k, k \in I$ $\theta \neq \pi k, k \in I$ 2

3. NPV's $\theta \neq \pi k, k \in I$ $\cot^2\theta$ 4. NPV's $B \neq \frac{\pi}{2} + \pi k, k \in I$ $B \neq \pi k, k \in I$ 1

5. NPV's $\theta \neq \frac{\pi}{2} + \pi k, k \in I$ $\theta \neq \pi k, k \in I$ $\tan^2\theta$

6. NPV's $x \neq \frac{\pi}{2} + \pi k, k \in I$ $x \neq \pi k, k \in I$ $\cot x$

7. NPV's $A \neq \pi k, k \in I$ $\sin A$ 8. No NPV's $\sin\theta$

6.2 Sum, Difference and Double Angle Identities

Goals:

- Using sum, difference, and double angle identities to determine exact values of trigonometric expressions.
- Using sum, difference, and double angle identities to simplify certain trigonometric expressions.

(I) Sum and difference formulae

Trigonometric relationships often involve angle measures that are either the sum or difference of other angle measures. These formulae include:

Sum Formulae:

$\sin(A + B) = \sin A \cos B + \cos A \sin B$

$\cos(A + B) = \cos A \cos B - \sin A \sin B$

$$\tan(A + B) = \frac{\tan A + \tan B}{1 - \tan A \tan B}$$

Difference Formulae:

$\sin(A - B) = \sin A \cos B - \cos A \sin B$

$\cos(A - B) = \cos A \cos B + \sin A \sin B$

$$\tan(A - B) = \frac{\tan A - \tan B}{1 + \tan A \tan B}$$

The advantage to using these formulae is to determine the exact value of angles. Each of these formulae are reversible.

Example: Simplify each expression using sum and difference formulae and evaluate.

(a) $\sin \overset{A}{48^\circ} \cos \overset{B}{18^\circ} - \cos \overset{A}{48^\circ} \sin \overset{B}{18^\circ}$

This expression is identical to the right side of which formulae ?

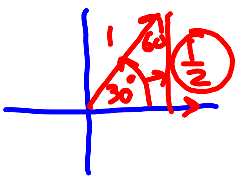
$\sin A \cos B - \cos A \sin B$

$= \sin(\underset{48^\circ}{A} - \underset{18^\circ}{B})$

$= \sin[48^\circ - 18^\circ]$

$= \sin(30^\circ)$

$= \frac{1}{2}$



(b) $\cos(255^\circ)$

$= \cos(\overset{A}{225^\circ} + \overset{B}{30^\circ})$

$= \cos(225^\circ)\cos(30^\circ) - \sin(225^\circ)\sin(30^\circ)$

$= \left(-\frac{\sqrt{2}}{2}\right)\left(\frac{\sqrt{3}}{2}\right) - \left(-\frac{\sqrt{2}}{2}\right)\left(\frac{1}{2}\right)$

$= -\frac{\sqrt{6}}{4} + \frac{\sqrt{2}}{4}$

$= \frac{-\sqrt{6}+\sqrt{2}}{4}$

Which TWO angles can we ADD or SUBTRACT to produce 255° which would include ONE of the special angles (30°, 45° or 60°) ?

ie. $\cos(255^\circ) = \cos(225^\circ + 30^\circ)$ ✓

$255^\circ - 30^\circ = 225^\circ$

or $\cos 255^\circ = \cos(210^\circ + 45^\circ)$

Which formula for cosine should we use?

$\cos[A+B] = \cos A \cos B - \sin A \sin B$

0.5	0.707...	0.866
$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$

$\frac{23(180)}{12} = 345^\circ$

(c) $\tan\left(\frac{23\pi}{12}\right)$

We can convert $\frac{23\pi}{12}$ to degrees and then consider the TWO angles that can ADD or SUBTRACT to produce $\underline{345}$° which would include ONE of the special angles (30°, 45° or 60°) ?

ie. tan ($\underline{345}$°) = tan($\underline{300}$° + $\underline{45}$°)

Which formula for tangent should we use?

$\tan(A+B) = \frac{\tan A + \tan B}{1 - \tan A \tan B}$

$\boxed{\begin{array}{r} 345^\circ \\ -45^\circ \\ \hline 300^\circ \end{array}}$

$345^\circ = 300^\circ + 45^\circ$

$\tan(300^\circ + 45^\circ)$

$= \frac{\tan 300^\circ + \tan 45^\circ}{1 - \tan 300^\circ \tan 45^\circ}$

$= \frac{\frac{-\frac{\sqrt{3}}{2}}{\frac{1}{2}} + \frac{\sqrt{2}/2}{\sqrt{2}/2}}{1 - \left(\frac{-\frac{\sqrt{3}}{2}}{\frac{1}{2}}\right)(1)}$

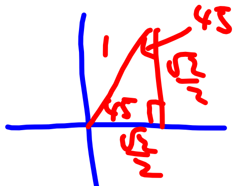
$= \frac{-\sqrt{3} + 1}{1 + \sqrt{3}}$

$= \frac{(1-\sqrt{3})(1-\sqrt{3})}{(1+\sqrt{3})(1-\sqrt{3})} = \frac{1 - 2\sqrt{3} + 3}{1 - 3}$

$= \frac{4 - 2\sqrt{3}}{-2} = -2 + \sqrt{3}$

$\div -2$

300° $\tan \angle = \frac{\sin \angle}{\cos \angle}$ 45



(II) Double angle formulae

P.306
1a, d
2b, d
5b
8a, b, f

Using double-angle identities we can determine exact trig ratios of angles that are multiples of 15°. The double –angle identities can be derived from the angle sum identities when B = A. These formulae are also reversible.

Double-Angle Identities:

sin 2A = 2sin A cos A

cos 2A = cos²A – sin²A

$\tan 2A = \frac{2\tan A}{1-\tan^2 A}$

Example: Simplify each expression using double – angle identities and evaluate.

(a) $\cos^2\left(\frac{\pi}{3}\right) - \sin^2\left(\frac{\pi}{3}\right)$ ← This expression is identical to the right side of which double – angle identity ? $\cos 2A = \cos^2 A - \sin^2 A$

$A = \frac{\pi}{3}$

$= \cos\left[2\left(\frac{\pi}{3}\right)\right]$

$= \cos\left[\frac{2\pi}{3}\right]$

$= -\frac{1}{2}$

(b) $\boxed{\dfrac{2\tan 105^\circ}{1-\tan^2 105^\circ}}$ ← This expression is identical to the right side of which double – angle identity ? $\tan 2A = \dfrac{2\tan A}{1-\tan^2 A}$

$A = 105^\circ$

$= \tan\left[2(105^\circ)\right]$

S | A
T | C

$= \tan\left[210^\circ\right]$

$= \dfrac{\sin 210^\circ}{\cos 210^\circ} = \dfrac{-1/2}{-\sqrt{3}/2} = -\dfrac{1}{\cancel{2}} \cdot -\dfrac{\cancel{2}}{\sqrt{3}} = \dfrac{1 \cdot \sqrt{3}}{\sqrt{3} \cdot \sqrt{3}} = \dfrac{\sqrt{3}}{3}$

(c) $2\sin\left(\frac{\pi}{12}\right)\cos\left(\frac{\pi}{12}\right)$ ← This expression is identical to the right side of which double – angle identity ? $\sin 2A = 2\sin A\cos A$

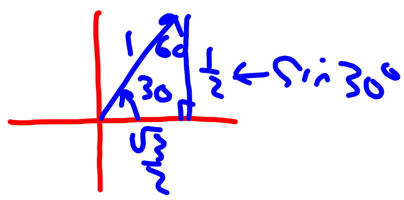
$A = \frac{\pi}{12}$

$= \sin\left[2\left(\frac{\pi}{12}\right)\right]$

$= \sin\left[\frac{\pi}{6}\right]$

$= \frac{1}{2}$

$\frac{\pi}{6} = \frac{180^\circ}{6} = 30^\circ$



(III) Equivalent formulae for cos 2A

Given the double – angle identity:

$\cos 2A = \cos^2 A - \sin^2 A$

Use the Pythagorean identity $\cos^2 A + \sin^2 A = 1$ to solve for $\sin^2 A$.

$\sin^2 A = 1 - \cos^2 A$

Substitute this result for $\sin^2 A$ into the cos 2A formula to derive an equivalent formula.

$\cos 2A = \cos^2 A - (1 - \cos^2 A)$

$\cos 2A = \cos^2 A - 1 + \cos^2 A$

$\boxed{\cos 2A = 2\cos^2 A - 1}$

Equivalent Formula for cos 2A
$\cos 2A = 2\cos^2 A - 1$

$\cos 2A = \cos^2 A - \sin^2 A$

Use the Pythagorean identity $\cos^2 A + \sin^2 A = 1$ to solve for $\cos^2 A$.

$\cos^2 A = 1 - \sin^2 A$

Substitute this result for $\cos^2 A$ into the cos 2A formula to derive an equivalent

$\cos 2A = 1 - \sin^2 A - \sin^2 A$

$\boxed{\cos 2A = 1 - 2\sin^2 A}$

Equivalent Formula for cos 2A
$\cos 2A = 1 - 2\sin^2 A$

(IV) Using sum, difference, and double – angle identities to simplify trig expressions

Simplifying Trig Expressions using Identities.

1. Given the expression $\dfrac{\sin 2x}{\boxed{1-\cos 2x}}$

(a) Determine the non – permissible roots algebraically.

NPV's

$1-\cos 2x \neq 0$

$-\cos 2x \neq -1$

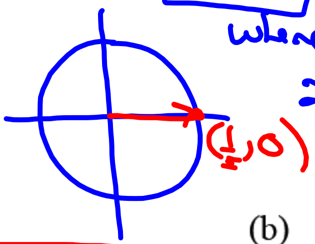
$\cos 2x \neq 1$

where

$2x \neq 0° + 360°n, n \in I$

$2x \neq 360°n, n \in I$

$x \neq 180°n, n \in I$



(b) Simplify the expression.

① Sub in appropriate double angle formulae so everything is in terms of sin x and cos x

$$\frac{\sin 2x}{1-\cos 2x} \rightarrow \frac{2\sin x\cos x}{1-(1-2\sin^2 x)}$$

$$= \frac{2\sin x\cos x}{\cancel{1}-\cancel{1}+2\sin^2 x}$$

$$= \frac{\cancel{2\sin x}\cos x}{2\sin^2 x} = \frac{\cos x}{\sin x} = \cot x$$

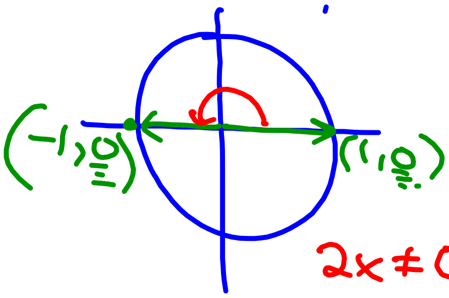
~~(c)~~ Using technology, verify the answer in (b) when $0 \le x \le 2\pi$.

2. Given the expression $\frac{\cos 2x + sin^2 x}{\sin 2x}$

(a) Determine the non – permissible roots algebraically.

$\sin 2x \neq 0$

where ?



$2x \neq 0 + \pi n, n \in I$

$2x \neq \pi n, n \in I$

$\boxed{x \neq \frac{\pi n}{2}, n \in I}$

(b) Simplify the expression.

Use

$\cos 2x = \cos^2 x - \sin^2 x$

$\frac{\cos 2x + \sin^2 x}{\sin 2x} \rightarrow \frac{\cos^2 x - \cancel{\sin^2 x} + \cancel{\sin^2 x}}{2 \sin x \cos x}$

$\frac{1 \cos^{\cancel{2}} x}{2 \sin x \cancel{\cos x}}$

$= \frac{1}{2} \cdot \frac{\cos x}{\sin x}$

$= \frac{1}{2} \cot x$ or $\frac{\cot x}{2}$

3. Given the expression $\frac{1-\cos 2x+\sin 2x}{1+\cos 2x+\sin 2x}$

(a) Determine the non – permissible roots algebraically.

(b) Simplify the expression.

(V) Using sum, difference, and double – angle identities to determine an exact value

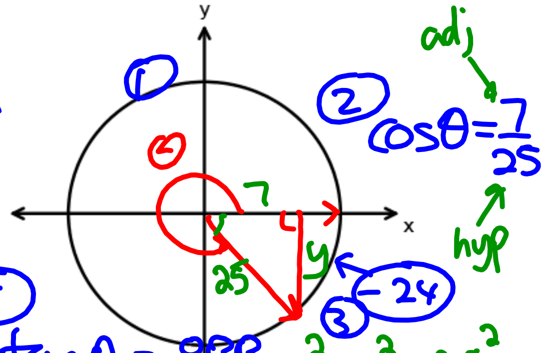
Example:

4th quad ∠

If $\cos\theta = \frac{7}{25}$ where $270° \le \Theta \le 360°$ then determine the exact values of:

(a) tan 2Θ

need double ∠ formula for tan



④ $\tan 2\theta = \frac{2\tan\theta}{1-\tan^2\theta}$

⑤ $\tan\theta = \frac{opp}{adj}$

$\tan\theta = -\frac{24}{7}$

$7^2 + y^2 = 25^2$

$49 + y^2 = 625$

$y^2 = 576$

$y = -\sqrt{576}$

$y = -24$

⑥ $\tan 2\theta = \frac{2\left(-\frac{24}{7}\right)}{1-\left(-\frac{24}{7}\right)^2}$

LCD = 49

$= \frac{-\frac{48}{7}\cdot 49}{49\cdot 1 - \frac{576}{49}\cdot 49} = \frac{-336}{49-576} = \frac{-336}{-527} = \frac{336}{527}$

(b) $\sin\left(\theta + \frac{3\pi}{2}\right)$

need sin Addition formula

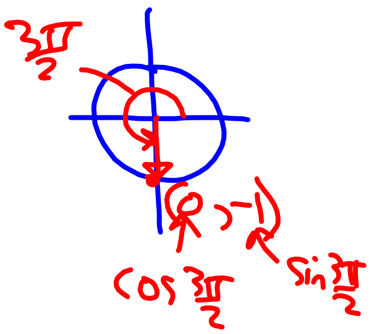
$\sin\left(\theta + \frac{3\pi}{2}\right)$

$= \sin\theta\cos\left(\frac{3\pi}{2}\right) + \sin\left(\frac{3\pi}{2}\right)\cos\theta$

$= \sin\theta \times (0) + (-1)\cos\theta$

$= 0 - 1\times\left(\frac{7}{25}\right)$

$= -\frac{7}{25}$



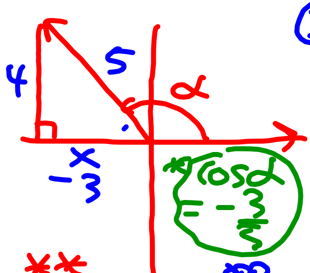
(VI) Using a Trig Equation to Determine the Exact Value of a Trig Expression

Example: 2ND Quad

If $\frac{\pi}{2} < \alpha < \beta < \pi$ and $\sin\alpha = \frac{4}{5}$ and $\tan\beta = -\frac{3}{4}$ then determine

(a) $\cos(\alpha + \beta)$. (b) $\tan(\alpha + \beta)$.

① α



$\boxed{\sin\alpha = \frac{4}{5}}$ ← opp, ← hyp

$x^2 + 4^2 = 5^2$

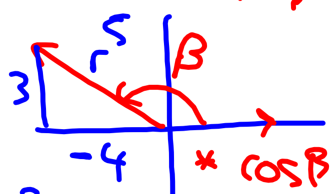
$\vdots$

$x = -3$

$\cos\alpha = -\frac{3}{5}$

② β

$\tan\beta = -\frac{3}{4}$ ← opp, ← adj



$3^2 + (-4)^2 = r^2$

$\vdots$

$r = 5$

$\cos\beta = -\frac{4}{5}$

$\sin\beta = \frac{3}{5}$

③ $\cos(\alpha + \beta)$

$= \cos\alpha\cos\beta - \sin\alpha\sin\beta$

$= \left(-\frac{3}{5}\right)\left(-\frac{4}{5}\right) - \left(\frac{4}{5}\right)\left(\frac{3}{5}\right)$

$= \frac{12}{25} - \frac{12}{25}$

$= 0$

Set P.306 – 308 #1 - #8, #10 - #12, #14 - #20, C1

Worksheet – Proving Trig Identities

Formulae

Key Ideas

- You can use the sum and difference identities to simplify expressions and to determine exact trigonometric values for some angles.

Sum Identities

$\sin(A + B) = \sin A \cos B + \cos A \sin B$

$\cos(A + B) = \cos A \cos B - \sin A \sin B$

$\tan(A + B) = \dfrac{\tan A + \tan B}{1 - \tan A \tan B}$

Difference Identities

$\sin(A - B) = \sin A \cos B - \cos A \sin B$

$\cos(A - B) = \cos A \cos B + \sin A \sin B$

$\tan(A - B) = \dfrac{\tan A - \tan B}{1 + \tan A \tan B}$

- The double-angle identities are special cases of the sum identities when the two angles are equal. The double-angle identity for cosine can be expressed in three forms using the Pythagorean identity, $\cos^2 A + \sin^2 A = 1$.

Double-Angle Identities

$\sin 2A = 2 \sin A \cos A$

$\cos 2A = \cos^2 A - \sin^2 A$
$\cos 2A = 2\cos^2 A - 1$
$\cos 2A = 1 - 2\sin^2 A$

$\tan 2A = \dfrac{2 \tan A}{1 - \tan^2 A}$

$\sin^2\theta + \cos^2\theta = 1$ $\cot^2\theta + 1 = \csc^2\theta$ $1 + \tan^2\theta = \sec^2\theta$

1. Calculate the exact value of each expression.

a) $\sin 75^0 \cos 15^0 + \cos 75^0 \sin 15^0$

b) $\cos\dfrac{5\pi}{12}\cos\dfrac{\pi}{12} - \sin\dfrac{5\pi}{12}\sin\dfrac{\pi}{12}$

c) $\cos 75^0$

d) $\sin\dfrac{11\pi}{12}$

e) $\dfrac{\tan\frac{2\pi}{3} + \tan\frac{\pi}{12}}{1 - \tan\frac{2\pi}{3}\tan\frac{\pi}{12}}$

(f) $1 - 2\sin^2(15°)$

(g) $4\sin\left(\frac{\pi}{8}\right)\cos\left(\frac{\pi}{8}\right)$

2. Simplify completely.

a) $\cos \overset{A}{2x} \cos \overset{B}{x} + \sin \overset{A}{2x} \sin \overset{B}{x}$

b) $\sin(\overset{A}{30^0} + \overset{B}{x}) + \sin(\overset{A}{30^0} - \overset{B}{x})$

c) $\cos x \cos y(\tan x + \tan y)$

d) $2\cos^2 10^0 - 1$

e) $2\sin 35^0 \cos 35^0$

f) $\dfrac{2\tan 25^0}{1-\tan^2 25^0}$

g) $\cos\left(\frac{3\pi}{2}+\theta\right) + \cos\left(\frac{3\pi}{2}-\theta\right)$

h) $(1 - \sin^2 x)(1 - \tan^2 x)$

i) $\sin x \tan x + \cos 2x \sec x$

j) $\dfrac{1+\cos 2x}{\cot x}$

3. a) Prove $\cos\left(\frac{\pi}{2}+\theta\right) = -\sin\theta$

b) Prove $\sin(\pi + x) = -\sin x$

4. If $\sin\alpha = \frac{3}{5}$ and $\sin\beta = \frac{24}{25}$ and if $\underbrace{0 < \alpha < \frac{\pi}{2}}_{\text{First Quad}}$ $\boxed{< \beta < \pi}$ then determine $\sin(\alpha + \beta)$.

2nd Quad

$= \sin\alpha \cos\beta - \sin\beta \cos\alpha$

ANSWERS:

1.(a) 1 (b) 0 (c) $\frac{\sqrt{6}-\sqrt{2}}{4}$ (d) $\frac{\sqrt{6}-\sqrt{2}}{4}$ (e) -1 (f) $\frac{\sqrt{3}}{2}$ (g) $\sqrt{2}$

2(a) cos x (b) cos x (c) sin(x + y) (d) cos 20° (e) sin 70° (f) tan 50°
(g) 0 (h) cos 2x (i) cos x (j) sin 2x

4. $\frac{3}{5}$

6.3 Proving Identities

Goal:

- Using reciprocal identities, Pythagorean identities, sum and difference identities, and double angle identities to prove other trigonometric identities.

(I) Verify versus Proving Identities

To prove identities, one or both sides of the identity must be rewritten in terms so that both sides are identical. There can be no algebraic interaction between the two sides.

Strategies for validating an identity include:

- Writing expressions in terms of sine and cosine
- Expressing the given trig functions in terms of a single trig function (sin x or cos x)
- Factoring expressions
- Writing expressions with a common denominator
- Expanding an expression, such as multiplying two binomials
- Writing one fraction as two or more fractions
- Multiplying by the conjugate
- Multiplying an expression by a fraction equivalent to 1.

Proving Identities

Example: Prove the given equation is an identity and identify the non – permissible roots.

L.H.S R.H.S

$\cot^2x+1=\csc^2x$

$$(1+\cot^2x)(1-\cos 2x)=2$$

$\cos 2x = 1-2\sin^2x$

LHS	RHS
$\csc^2x[1-(1-2\sin^2x)]$ $\frac{1}{\sin^2x}[\cancel{1}-\cancel{1}+2\sin^2x]$ $\frac{1}{\cancel{\sin^2x}}\cdot\frac{[2\cancel{\sin^2x}]}{1}$ 2	2

Strategies :

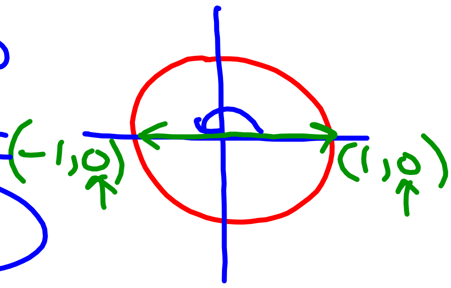
- Express in terms of sine and cosine
- Use quotient and double angle identities
- Can we use a Pythagorean identity?
- Perform the algebraic operations indicated

Non – permissible roots: $\sin^2x\neq 0$

$\sin x \neq 0$

$x\neq 0+\pi n, n\in I$

$x\neq \pi n, n\in I$



Verifying Identities

Example: Verify the given identity both numerically.

$$(1 + \cot^2 x)(1 - \cos 2x) = 2$$

Numerically

Verify when x = 45°

LHS	RHS

Example: Prove the given equation is an identity and identify the non – permissible roots.

$$\cos 2\theta = \frac{1-\tan^2\theta}{1+\tan^2\theta}$$

$\underbrace{1+\tan^2\theta}$ $= \sec^2\theta$

LHS	RHS
$\cos 2\theta$	$\frac{1-\tan^2\theta}{1+\tan^2\theta}$ $\frac{1-\frac{\sin^2\theta}{\cos^2\theta}}{\sec^2\theta}$ $\frac{1-\frac{\sin^2\theta}{\cos^2\theta}}{\frac{1}{\cos^2\theta}}$ $\left(1-\frac{\sin^2\theta}{\cos^2\theta}\right)\cos^2\theta$ $\underbrace{\cos^2\theta - \sin^2\theta}$ $\cos 2\theta$

* Non – permissible roots:

$1+\tan^2\theta \neq 0$

$\tan^2\theta \neq -1$

$\tan\theta \neq \sqrt{-1}$

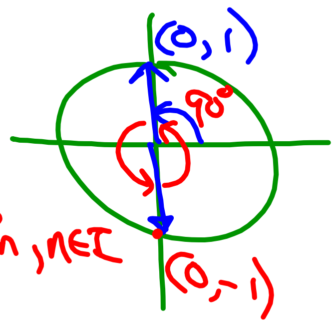
no solution

$\cos^2\theta \neq 0$

$\underbrace{\cos\theta \neq 0}$

$x = ?$

$x \neq 90° + 180°n, n \in I$



Example: Prove the given equation is an identity and identify the non – permissible roots.

L.H.S R.H.S

1. $\dfrac{1-\cos x}{\boxed{\sin x}} = \dfrac{\sin x}{\boxed{1+\cos x}}$

Strategy: $(a+b)(a-b)$

- Use conjugates. (ie. multiple one binomial by its conjugate) to rewrite the denominator or numerator in terms of sin x

NPV's
$\sin x \neq 0$
$(-1,0)$ $(1,0)$
$x \neq 180^\circ n, n \in I$

$1+\cos x \neq 0$
$\cos x \neq -1$
$x \neq 180^\circ + 360^\circ n, n \in I$

$\dfrac{\sin x}{(1+\cos x)}\cdot\dfrac{(1-\cos x)}{(1-\cos x)}$

$\dfrac{\sin x(1-\cos x)}{1-\cos^2 x}$

$\dfrac{\cancel{\sin x}(1-\cos x)}{\sin^{\cancel{2}} x}$

$\dfrac{1-\cos x}{\sin x}$

Remember
$(a-b)(a+b)$
$= a^2 - b^2$
$\boxed{1-\cos^2 x = \sin^2 x}$

2. $\dfrac{1}{1+\sin x}\cdot\dfrac{(1-\sin x)}{(1-\sin x)} = \dfrac{\sec x - \sin x \sec x}{\cos x}$

Strategy:

- Remove a common factor and use conjugates

GCF = Sec x

$\dfrac{1-\sin x}{1-\sin^2 x}$

$\dfrac{1-\sin x}{\cos^2 x}$

$\dfrac{\sec x(1-\sin x)}{\cos x}$

$\dfrac{\frac{1}{\cos x}(1-\sin x)}{\cos x}$

$\dfrac{\frac{1-\sin x}{\cos x}}{\cos x}$

$\dfrac{1-\sin x}{\cos x}\times\dfrac{1}{\cos x}$

$\dfrac{1-\sin x}{\cos^2 x}$

3. $\cot x - \csc x = \dfrac{\boxed{\cos 2x} - \cos x}{\boxed{\sin 2x} + \sin x}$

Strategy:

➢ When we use our double – angle identities it may lead to expressions that require factoring.

$\dfrac{\cos x}{\sin x} - \dfrac{1}{\sin x}$

$\dfrac{\cos x - 1}{\sin x}$

rearrange

$\dfrac{2\cos^2 x - 1 - \cos x}{2\sin x\cos x + \sin x}$

GCF = $\sin x$

FACTOR

$\dfrac{2\cos^2 x - \cos x - 1}{\sin x[2\cos x + 1]}$

$\dfrac{\cancel{(2\cos x + 1)}(\cos x - 1)}{\sin x\,[\cancel{2\cos x + 1}]}$

$\dfrac{\cos x - 1}{\sin x}$

$\sin 2x = 2\sin x\cos x$

$\cos 2x = \cos^2 x - \sin^2 x$

$\cos 2x = \boxed{2\cos^2 x - 1}$ *

$\cos 2x = 1 - 2\sin^2 x$

$2\cos^2 x - \cos x - 1$

let $m = \cos x$

$2m^2 - 1m - 1$

$(2m + 1)(m - 1)$

$+1m$

$-2m$

$(2\cos x + 1)(\cos x - 1)$

Chapter 6 Trigonometric Identities 32

Key Ideas

- Verifying an identity using a specific value validates that it is true for that value only. Proving an identity is done algebraically and validates the identity for all permissible values of the variable.
- To prove a trigonometric identity algebraically, separately simplify both sides of the identity into identical expressions.
- It is usually easier to make a complicated expression simpler than it is to make a simple expression more complicated.
- Some strategies that may help you prove identities include:
  - Use known identities to make substitutions.
  - If quadratics are present, the Pythagorean identity or one of its alternate forms can often be used.
  - Rewrite the expression using sine and cosine only.
  - Multiply the numerator and the denominator by the conjugate of an expression.
  - Factor to simplify expressions.

Practice question and prescribed work

Prove the identity $\frac{\boxed{\sin 2x} - \cos x}{4\sin^2 x - 1} = \frac{\sin^2 x \cos x + \cos^3 x}{2\sin x + 1}$

or all permissible values of x.

GCF = cosx

$$\frac{2\sin x\cos x - \cos x}{4\sin^2 x - 1}$$

FACTOR $a^2 - b^2 = (a-b)(a+b)$

$$\frac{\cos x(\cancel{2\sin x - 1})}{(\cancel{2\sin x - 1})(2\sin x + 1)}$$

$$\frac{\cos x}{2\sin x + 1}$$

GCF = cosx

$$\frac{\cos x(\sin^2 x + \cos^2 x)}{2\sin x + 1}$$

$$\frac{\cos x}{2\sin x + 1}$$

P.314- 315#1 - #8, #10 - #17, C2, C3

Worksheet – Proving Trig Identities

1. $\dfrac{1}{\sin x\cos x}-\dfrac{\cos x}{\sin x}=\tan x$

2. $\dfrac{\cot x(1+\tan^2 x)}{\tan x}=\csc^2 x$

3. $\tan^2 x-\sin^2 x=\tan^2 x\sin^2 x$

4. $\dfrac{1-3\cos x-4\cos^2 x}{\sin^2 x}=\dfrac{1-4\cos x}{1-\cos x}$

5. $\dfrac{\sin x}{1+\cos x}=\dfrac{1-\cos x}{\sin x}$

6. $\dfrac{\cos x}{\sec x-1}-\dfrac{\cos x}{\tan^2 x}=\cot^2 x$

7. $\csc x\cos^2 x+\sin x=\csc x$

8. $\sin x+\cot^2 x\sin x=\csc x$

9. $\cot x+\tan x=\csc x\sec x$

10. $\sec^2 x+\tan^2 x\sec^2 x=\sec^4 x$

11. $\dfrac{\sec x}{\sin x}-\dfrac{\sin x}{\cos x}=\cot x$

12. $\dfrac{1-\sin^2 x}{1+\cot^2 x}=\sin^2 x\cos^2 x$

13. $\dfrac{\tan^2 x}{1+\tan^2 x}=\sin^2 x$

14. $\dfrac{\tan x+\sin x}{1+\cos x}=\tan x$

15. $\dfrac{\sec x+\tan x}{\dfrac{1}{1-\sin x}}=\cos x$

16. $\sec x+\csc x=\dfrac{1+\tan x}{\sin x}$

17. $\sec x+\dfrac{1}{\cot x}=\dfrac{1+\sin x}{\cos x}$

18. $(\cot A+\tan A)^2=\csc^2 A\sec^2 A$

19. $\dfrac{\tan x}{1+\sec x}+\dfrac{1+\sec x}{\tan x}=2\csc x$

20. $\dfrac{\sec^4 x-1}{\tan^2 x}=\tan^2 x+2$

21. $\dfrac{\cos x-\tan x}{\sin x\cos x}=\csc x-\sec^2 x$

6.4 Solving Trigonometric Equations Using Identities

Goals:

- Solve, algebraically and graphically, first and second degree trig equations with the domain expressed in degrees and radians using known identities.
- Identifying and correcting errors in a solution for a trig equation.

(I) Solving a Trigonometric Equation

Graphically

Example: Graphically determine the solutions for

$\sin 2x = \sqrt{3}\cos x$ where $0° \le x \le 360°$.

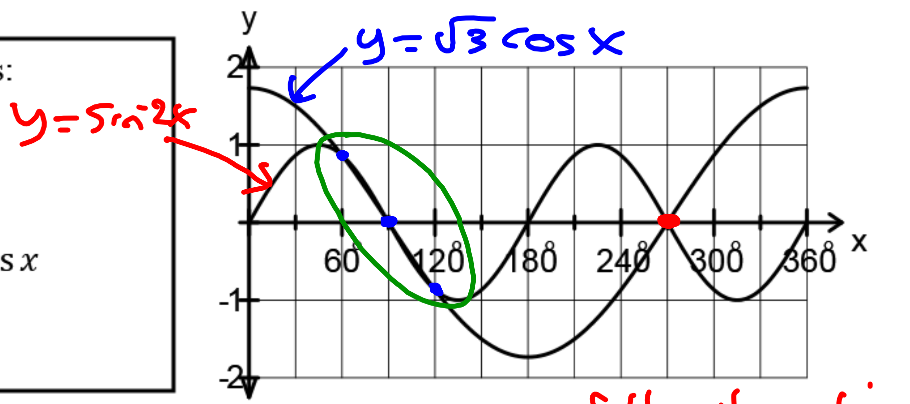
Graph the functions:

$y = \sin 2x$

and

$y = \sqrt{3}\cos x$

on the same grid.



NOTE: The solutions represent the x-coordinates of the intersection points

Solutions: $x = 270°, 60°, 90°, 120°$

Algebraically

Example: Algebraically determine the general solution for

$$\sin 2x = \sqrt{3}\cos x.$$

$\sin 2x = \sqrt{3}\cos x$

$2\sin x\cos x = \sqrt{3}\cos x$

$2\sin x\cos x - \sqrt{3}\cos x = 0$

GCF = $\cos x$

$\cos x(2\sin x - \sqrt{3}) = 0$

Apply zero product property $a \cdot b = 0$

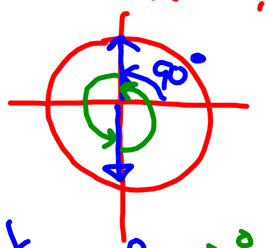
$a = 0 \mid b = 0$

To Solve a Trig Equation Involving Identities:

- Substitute a known identity to replace $\sin 2x$ = $2\sin x\cos x$
- Re-arrange the equation so that it is equal to 0.
- Consider factoring to solve the given equation.

$\cos x = 0$

where $x = 0$?

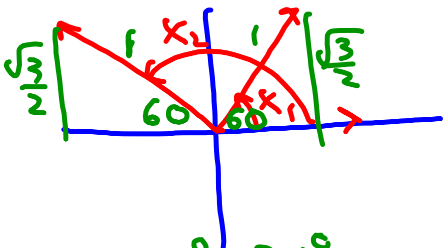


$x = 90° + 180°n, n \in I$

$2\sin x - \sqrt{3} = 0$

$2\sin x = \sqrt{3}$

$\sin x = \frac{\sqrt{3}}{2}$



$x_1 = 60° + 360°n, n \in I$

$x_2 = 120° + 360°n, n \in I$

Example: Determine the general solution for: $sin^2 x = \frac{1}{2}\tan x \cos x$

$\sin^2 x = \frac{1}{2}\boxed{\tan x}\cos x$

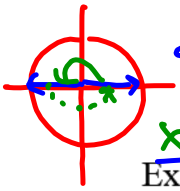
$\sin^2 x = \frac{1}{2}\frac{\sin x}{\cancel{\cos x}}\cancel{\cos x}$

$\sin^2 x = \frac{1}{2}\sin x$ ⟷ $\boxed{\text{rearrange so } = 0}$

$2\left[\sin^2 x - \frac{1}{2}\sin x = 0\right]$

$2\sin^2 x - \sin x = 0$ ← Factor GCF = sinx out

$\sin x\left[2\sin x - 1\right] = 0$ → $2\sin x - 1 = 0$

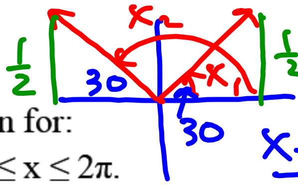


$\sin x = 0$

$y = 0$?

$x = 180°n, n \in I$

$\sin x = \frac{1}{2}$ S|A T|C



$x_1 = 30° + 360°n, n \in I$

$x_2 = 150° + 360°n, n \in I$

Example: Determine the exact solution for:
$1 + \cos 2x = \cos x$ where $0 \leq x \leq 2\pi$.

$\cancel{1} + 2\cos^2 x - \cancel{1} = \cos x$

$2\cos^2 x = \cos x$

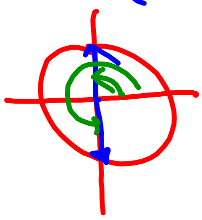
$2\cos^2 x - \cos x = 0$

$\cos x\left[2\cos x - 1\right] = 0$

Which identity should be used as a substitution for: $\cos 2x = 2\cos^2 x - 1$

$\cos x = 0$

$x = 0$?



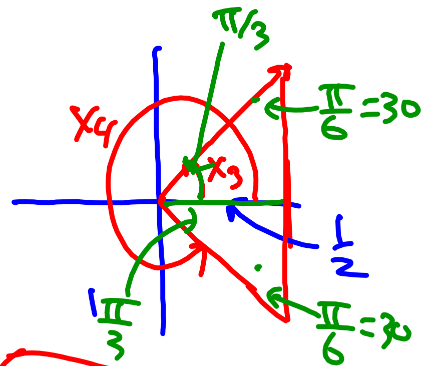
$x_1 = \frac{\pi}{2}$ $x_2 = \frac{3\pi}{2}$

$2\cos x - 1 = 0$

$\cos x = \frac{1}{2}$ S|A T|C

$x_3 = \frac{\pi}{3}$

$x_4 = \frac{5\pi}{3}$



P.320 #1b,d #2d #3a

Solving Trig Equations for Non – Special Angles

Example: Solve $\underbrace{\cos 2x}_{} + \sin^2 x = 0.7311$ where $\boxed{0° \le x \le 360°.}$

$\cos 2x$
$= \cos^2 x - \sin^2 x$

$$\cos^2 x - \cancel{\sin^2 x} + \cancel{\sin^2 x} = 0.7311$$

$$\cos^2 x = 0.7311$$

$$\cos x = \pm\sqrt{0.7311}$$

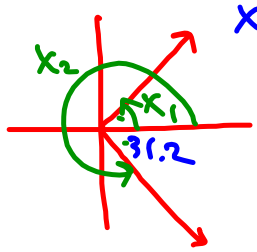
$$\cos x = \pm 0.855$$

$\cos x = 0.855$

S A
T C

$x = \cos^{-1}(0.855)$

$x = 31.2°$



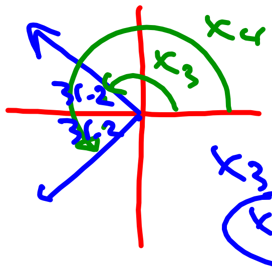
$x_1 = 31.2°$

$x_2 = 360° - 31.2°$

$x_2 = 328.8°$

$\cos x = -0.855$

S A
T C



$x_3 = 180° - 31.2°$

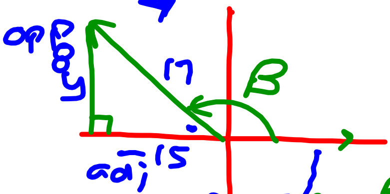
$x_3 = 148.8°$

$x_4 = 180 + 31.2°$

$x_4 = 211.2°$

Solving Trig Equations for Non – Special Angles

If $\tan\alpha = \frac{12}{5}$ and $\boxed{\cos\beta = -\frac{15}{17}}$, and if $\underbrace{\frac{\pi}{2} < \beta < \pi}_{\text{2ND}} < \alpha < \frac{3\pi}{2}$ (3rd), then determine:



$y^2 + (-15)^2 = 17^2$ $\cos\beta = -\frac{15}{17}$ (adj / hyp)

$y = 8$

$\tan\beta = -\frac{8}{15}$

(a) $\tan(\alpha + \beta)$.

$$= \frac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta}$$

$$= \frac{\frac{12}{5} + \left(-\frac{8}{15}\right)}{1 - \left(\frac{12}{5}\right)\left(-\frac{8}{15}\right)} = \frac{\frac{12}{5} - \frac{8}{15}}{\frac{1}{1} + \frac{96}{75}}$$

LCD = 75

$$= \frac{12(15) - 8(5)}{75 + 96} = \frac{180 - 40}{171}$$

$$= \frac{140}{171}$$

(b) $\cos 2\alpha - \sin 2\beta$

$$\cos^2\alpha - \sin^2\alpha - 2\sin\beta\cos\beta$$

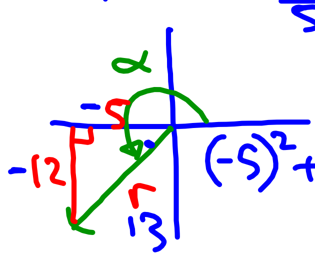
$$= \left(-\frac{5}{13}\right)^2 - \left(-\frac{12}{13}\right)^2 - 2\left(\frac{8}{17}\right)\left(-\frac{15}{17}\right)$$

$$= \frac{25}{169} - \frac{144}{169} + \frac{240}{289}$$

Stop

NOTE
from above diagra.
$\sin\beta = \frac{8}{17}$

$\tan\alpha = \frac{12}{5}$ ← opp / ← adj



$(-5)^2 + (-12)^2 = r^2$

$r = 13$

$\sin\alpha = -\frac{12}{13}$

$\cos\alpha = -\frac{5}{13}$

(II) Identifying and Correcting Errors in a Solution

Analyze the solution to the trigonometric equation and identify the error. Provide the correct solution.

Example: Solve $\sin^2 x - \sin x = 0$ where $\boxed{0° \le x \le 360°}$

This is a Quadratic Equation

$\sin^2 x - \sin x = 0$

Step I $\sin^2 x = \sin x$

Error

Step II $\dfrac{\sin^2 x}{\sin x} = \dfrac{\sin x}{\sin x}$

Division by sin x creates a linear statement

Step IIII $\sin x = 1$

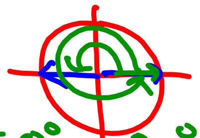
Step IV $x = 90°$

Correct Solution

$\sin^2 x - \sin x = 0$

$\sin x(\sin x - 1) = 0$

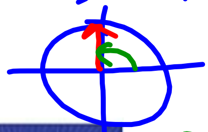
$\underbrace{\sin x = 0}_{y = 0?}$



$x = 0°, 180°, 360°$

$\sin x - 1 = 0$

$\underbrace{\sin x = 1}_{y = 1?}$



$x = 90°$

Key Ideas

- Reciprocal, quotient, Pythagorean, and double-angle identities can be used to help solve a trigonometric equation algebraically.
- The algebraic solution for a trigonometric equation can be verified graphically.
- Check that solutions for an equation do not include non-permissible values from the original equation.
- Unless the domain is restricted, give general solutions. For example, for $2 \cos x = 1$, the general solution is $x = \frac{\pi}{3} + 2\pi n$ and $x = \frac{5\pi}{3} + 2\pi n$, where $n \in I$. If the domain is specified as $0° \le x < 360°$, then the solutions are 60° and 300°.

1b, d 2d 3a

P.320 #1 – #12, #14 – #16, #19

P.320 – P.320 #5, #6 9c